Lag-Bipartite Consensus of Linear Multi-agent Systems over Signed Directed Graph

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***Abstract*—This work focusses on studying lag-bipartite consensus for linear multi-agent systems (MAS) with a leader where the agents interact via a signed directed graph. In this consensus problem over a signed graph, one group of followers achieve the exact leader state value after a certain delay while another group achieves this value in magnitude, but opposite in sign after the same delay. The consensus is achieved by designing both distributed state feedback and distributed observer-based output feedback control protocols with appropriate design parameters. It is established in this work that if the underlying communication graph has a rooted directed spanning tree whose root is the leader, then the objective of realizing lag-bipartite consensus is achieved in this setting. The numerical examples verify the theoretical results of the work.**

***Index Terms*—Multi-agent systems, lag-bipartite consensus, signed directed graph, output feedback.**

I. Introduction

Over the years, MAS has evolved as a burgeoning research topic that can be ascribed to its implementations in numerous areas like unmanned aerial vehicles (UAV), robotics, smart grid, opinion dynamics [1], etc. Among all the collective behaviours of MAS [2], consensus is of particular interest for the control research community where all the networked agents agree upon a common value of a particular quantity of interest through their local interaction with one-another. When these agents collaborate or cooperate through the local control laws to realize the control objectives, it is known as collaborative or cooperative control. There is copious amount of studies carried out on collaborative control of MAS that interact over unsigned graph [3] where the agents achieve the control tasks by cooperating with one-another. Interestingly, such graphs fail to represent interactions in some practical scenarios where the objectives or interest of some of the agents may be different. In such cases, it is more plausible to assume that both cooperation and competition coexist in the MAS; examples include opinion dynamics [4], PageRank algorithm [5], two-party robotic vehicle systems [6], etc. Signed graphs [7] are more suitable to describe such interactions where a positive edge-weight represents cooperation or friendship between two neighbours and a negative edge-weight represents competition or enmity. This type of consensus study, therefore, is the main focus of this work.

In [8] bipartite consensus is first introduced over a signed graph where it needs to maintain structural balance property. The equivalence of consensus problem and bipartite consensus problem is discussed in [9]. Bipartite consensus of input saturated linear leaderless MAS is investigated in [6]. Mean-square bipartite consensus is investigated in [10] for first-order integral MAS subjected to measurement noise. The problem of sign-consensus have been studied for first order integral MAS and general linear MAS in [11] and [12], respectively. Quantized leaderless bipartite consensus for single integrator models has been studied in [13].

Time-delays has certain effects in communication network of agents that can never be ruled out. More specifically, state or input delays have some effects on consensus convergence that is investigated in [14], [15]. However, [16] introduces lag consensus where the followers' states lag behind the leader's corresponding state for a certain time. The study can indeed be helpful for network or traffic congestion problems [17]. It is noteworthy to mention that in the previous works of the authors' [18], [19], lag-bipartite consensus problem with adaptive coupling and with actuator saturation, respectively have been investigated over signed undirected graph with state feedback laws. However, directed graph (digraph) is more generalized and cost-effective than undirected graph as bidirectional connectivity is not required to be maintained in the former. Moreover, output feedback control laws have more advantages compared to the state feedback laws when it comes to practical modelling of control systems. Therefore, in this work, the focus is achieving this type of consensus on digraph and with suitable feedback law.

In this work, lag-bipartite consensus is studied over a weighted signed digraph. This type of consensus refers to a set of followers agreeing with the leader's state value in both magnitude and sign after a certain delay while the other set reaching agreement only on the magnitude and opposite in sign after the same delay. However, it is considered that only a few followers called pinning agents receive the direct information about the leader state value. Therefore, in order for the leader information to be accessible by all the follower agents, a rooted directed spanning tree whose root is at the leader node is considered in the communication graph topology. The contribution of this work are:

(i) First, this work studies lag-bipartite consensus for more general linear dynamical agents in leader-follower setting compared to lower order agents like single-integrator or double integrator models in leaderless MAS.
(ii) Second, the underlying communication graph considered is a signed digraph which is more generalized and feasible than an undirected graph from a practical implementation standpoint.
(iii) Third, the work in addition to proposing a distributed state feedback control protocol also proposes a distributed observer-based output feedback law, considering that only output measurements are accessible by neighbouring agents. This approach bears significant practical relevance [20].

The organization of the remaining portion of the paper is as follows. In Section II, the preliminaries on the used notations and the pertinent graph theory concepts are given. The MAS model and problem formulation have been provided in Section III. The principal technical results are laid out in Section IV. Section V discusses the simulation results and Section VI provides the concluding comments.

II. Preliminaries

This section provides the used notations and discuses some pertinent graph theory concepts.

A. Used Notations

Let, $\Im_N = \{1, 2, \ldots, N\}$. The N-dimensional Euclidean space and the set of all $M \times N$ real matrices are represented by $\mathbb{R}^N$ and $\mathbb{R}^{M\times N}$, respectively. Superscript T represents the transpose of a matrix or a vector. $P > 0$ ($P \geq 0$) and $P < 0$ ($P \leq 0$) represent a positive (semi-)definite and a negative (semi-)definite matrix, respectively. The i^{th} eigenvalue of a real square matrix P is denoted by $\lambda_i(P)$ while its minimum eigenvalue is denoted by $\lambda_{\min}(P)$. Identity matrix of $n \times n$ dimension is represented by I_n while $\mathbf{0}$ represents a zero vector or zero matrix of appropriate dimension. $\otimes$ represents the Kronecker product. $\text{diag}(g_1, g_2, \ldots, g_n)$ and $\text{col}(g_1, g_2, \ldots, g_n)$ represent a diagonal matrix and a column vector, respectively where g_i ($i \in \Im_n$) are diagonal and column elements, respectively. Signum function is denoted by *sign*. Re(.) represents the real part of a complex scalar. $||$ and $||.||$ denote the absolute value of a scalar and Euclidean norm of a vector, respectively. Null set is denoted by $\varnothing$.

B. Graph Theory

A weighted signed digraph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$ is considered in this paper for the N networked agents where $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$, $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$, and $\mathcal{A}_s = [a_{ij}]_{N\times N}$ are the node set, the edge set, and the weighted signed adjacency matrix, respectively. If $a_{ij} \neq 0$, then $(v_j, v_i) \in \mathcal{E}$; $a_{ij} = 0$, otherwise. No self-loop is considered in this work, i.e., $a_{ii} = 0$. Moreover, $a_{ij} \geq 0$ and $a_{ij} \leq 0$ represent the cooperative and the competitive interactions between two interacting agents, respectively. The graph is considered digon sign-symmetric, i.e., $a_{ij}a_{ji} > 0$. $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$ denotes the set of all the adjacent (neighbour) nodes of i^{th} agent. A digraph contains a rooted directed spanning tree if there exists at least a root node such that it has a directed path to every other node in the graph. The augmented graph $\overline{G} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ contains the leader node as the 0^{th} agent, i.e., v_0 where $\overline{\mathcal{V}} = \mathcal{V} \cup \{v_0\}$ and $\overline{\mathcal{E}} \subseteq \{(v_i, v_j) : v_i, v_j \in \overline{\mathcal{V}}\}$. Pinning matrix is represented by $D_p = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_N\}$ where $\delta_i > 0$, if i^{th} follower has direct access to leader state information, otherwise $\delta_i = 0$, $\forall i \in \Im_N$. Laplacian matrix is given as $\mathcal{L}_s = \mathcal{D}_s - \mathcal{A}_s$ where $\mathcal{D}_s = \text{diag}(\sum_{j\in\mathcal{N}_1} |a_{1j}|, \sum_{j\in\mathcal{N}_2} |a_{2j}|, \ldots, \sum_{j\in\mathcal{N}_N} |a_{Nj}|)$ is the degree matrix. A matrix $H_s \in \mathbb{R}^{N\times N}$ is defined such that $H_s = \mathcal{L}_s + D_p$.

In [8], it is shown that to achieve bipartite consensus the underlying graph topology must be structurally balanced [7]. If the node set $\mathcal{V}$ of a signed graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$ can be divided into two sets of nodes $\mathcal{V}_1$ and $\mathcal{V}_2$ in such a way that $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$, and $a_{ij} \geq 0$, $\forall v_i, v_j \in \mathcal{V}_q$, (or $\mathcal{V}_r$), and $a_{ij} \leq 0$, $\forall v_i \in \mathcal{V}_q$, $\forall v_j \in \mathcal{V}_r$ where $q \neq r$, $q, r \in \{1, 2\}$, then the signed graph is said to be structurally balanced. A column vector S is defined such that $S = \text{col}(s_1, s_2, \ldots, s_N)$ where $s_i = 1$, $\forall v_i \in \mathcal{V}_q$ and $s_i = -1$, $\forall v_i \in \mathcal{V}_r$.

Assumption 1*:* The followers' communication graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$ is structurally balanced and the augmented graph $\overline{G} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ contains a rooted directed spanning tree that has the root at the leader node. This means that there is at least an agent i for which $\delta_i > 0$, $i \in \Im_N$.

Remark 1*:* Under Assumption 1, $\text{Re}(\lambda_i(H_s)) > 0$ ($\forall i \in \Im_N$), and $\mathcal{L}_s$ has a simple 0 eigenvalue and all its other eigenvalues have positive real parts.

III. System Model and Problem Statement

This section provides the system dynamics and the problem statement.

A. Agent Dynamics

The system dynamics of N networked homogeneous linear followers can be written as,

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t),\ \forall i \in \Im_N \\ y_i(t) &= Cx_i(t) \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}^m$, and $u_i(t) \in \mathbb{R}^p$ are the state, output, and control input vectors of i^{th} agent, respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{m\times n}$ are the system matrix, input matrix, and output matrix for the agents, respectively.

The system dynamics of the leader is written as,

$$\begin{aligned} \dot{x}_0(t) &= Ax_0(t) \\ y_0(t) &= Cx_0(t) \end{aligned} \tag{2}$$

where $x_0(t) \in \mathbb{R}^n$ and $y_0(t) \in \mathbb{R}^m$ are the state and output vectors of the leader, respectively.

Assumption 2*:* The pair (A, B) is stabilizable and the pair (A, C) is detectable which are required for the solvability of algebraic Riccati equation (ARE) (6) and the observer-based protocol, respectively.

The definition of lag-bipartite consensus is as follows, *Definition:* For MAS (1) and (2) with any initial value, lag-bipartite consensus is achieved with suitable feedback control protocols if,

$$\lim_{t\to\infty} ||x_i(t) - s_i x_0(t-\Delta)|| = 0, \quad \forall i \in \Im_N \tag{3}$$

where s_i is defined in Section II-B, and $\Delta > 0$ is communication delay between the leader and the followers. Moreover, lag-bipartite consensus error is given as,

$$e_{x_i}(t) = x_i(t) - s_i x_0(t-\Delta), \quad \forall i \in \Im_N \tag{4}$$

B. Problem Formulation

Find a distributed state and a distributed observer-based output feedback control law over a structurally balanced signed digraph such that, for any initial value of all the agents in the MAS comprising the followers (1) and a leader (2), they achieve lag-bipartite consensus (3), or equivalently, $\lim_{t\to\infty} ||e_x(t)|| = 0$, where $e_x(t) = [e_{x_1}^T(t)\ e_{x_2}^T(t) \dots e_{x_N}^T(t)]^T \in \mathbb{R}^{Nn}$ is the stacked error vector.

IV. Principal Technical Results

This section spells out the main results of the work.

A. Distributed State Feedback Control Protocol

For the followers (1), the following distributed state feedback control protocol is designed,

$$\begin{aligned} u_i(t) = -cK(\sum_{j\in\mathcal{N}_i} |a_{ij}|(x_i(t) - \textit{sign}(a_{ij})x_j(t)) \\ + \delta_i(x_i(t) - s_i x_0(t-\Delta))) \end{aligned} \tag{5}$$

where $c > 0$ is the scalar coupling gain and $K \in \mathbb{R}^{p\times n}$ is the feedback gain matrix, both of which will be designed later. The following Theorem is provided to realize the lag-bipartite consensus with distributed state feedback control law.

Theorem 1. *MAS* (1) *and* (2) *achieve lag-bipartite consensus with distributed state feedback protocol* (5) *over a structurally balanced signed digraph if* $(A - c\lambda_i(H_s)BK)$ $(\forall i \in \Im_N)$ *is Hurwitz, and designed parameters are* $c \geq \frac{1}{2Re(\lambda_{min}(H_s))}$ *and* $K = B^T Q$*, where* $Q \in \mathbb{R}^{n\times n}$ *is the positive definite solution to the following ARE:*

$$A^T Q + QA - QBB^T Q + I_n = 0 \tag{6}$$

Proof. In a compact form, (4) can be expressed as,

$$\dot{e}_x(t) = (I_N \otimes A)e_x(t) + (I_N \otimes B)u(t) \tag{7}$$

where $u(t) = [u_1^T(t)\ u_2^T(t) \dots u_N^T(t)]^T$ is the stacked control input vector. Noting that $\textit{sign}(a_{ij})s_j = s_i, \forall (v_j, v_i) \in \mathcal{E}$, $u_i(t)$ can be written from (5) as,

$$u_i(t) = -cK(\sum_{j\in\mathcal{N}_i} |a_{ij}|(e_{x_i}(t) - \textit{sign}(a_{ij})e_{x_j}(t)) + \delta_i e_{x_i}(t)) \tag{8}$$

Equivalently, (8) can also be expressed as,

$$u(t) = -(cH_s \otimes K)e_x(t) \tag{9}$$

where H_s is defined in Section II-B. Then, using (9), error dynamical system (7) can be expressed as,

$$\dot{e}_x(t) = (I_N \otimes A - cH_s \otimes BK)e_x(t) \tag{10}$$

Clearly, lag-bipartite consensus is reached when $(I_N \otimes A - cH_s \otimes BK)$ is Hurwitz. To that end, there exits a non-singular matrix $\Sigma \in \mathbb{R}^{N\times N}$ such that $\Sigma^{-1}H_s\Sigma = J = \text{diag}(J_1, J_2, \dots, J_l) \in \mathbb{R}^{N\times N}$ is the Jordan canonical form of H_s where J_k $(k \in \Im_l)$ are upper triangular Jordan blocks whose diagonal elements are $\lambda_i(H_s)$ $(i \in \Im_N)$. Let, $e_y(t) = (\Sigma^{-1}\otimes I_n)e_x(t)$ be the new error state transformation. Then, the new error state dynamics can be given as,

$$\begin{aligned} \dot{e}_y(t) &= (\Sigma^{-1} \otimes I_n)\dot{e}_x(t) \\ &= (I_N \otimes A - cJ \otimes BK)e_y(t) \end{aligned} \tag{11}$$

It can be easily observed that $(I_N \otimes A - cJ \otimes BK)$ is also an upper triangular block matrix having diagonal block elements $(A - c\lambda_i(H_s)BK)$ $(i \in \Im_N)$. Therefore, the eigenvalues of the matrix $(I_N \otimes A - cJ \otimes BK)$ are given by the eigenvalues of the matrix $(A - c\lambda_i(H_s)BK)$. Hence, if $(A - c\lambda_i(H_s)BK)$ are Hurwitz matrices, then it follows that lag-bipartite consensus is achieved. In addition, to design the scalar coupling gain and feedback gain matrix, the following is obtained using (6),

$$\begin{aligned} (A - c\lambda_i(H_s)BK)^T Q + Q(A - c\lambda_i(H_s)BK) \\ = -I_n - (2c\lambda_i(H_s) - 1)QBK \end{aligned}$$

Since $Q > 0$, then it is obvious from Lyapunov theory [20] that $(A - c\lambda_i(H_s)BK)$ $(\forall i \in \Im_N)$ is Hurwitz if the scalar coupling gain $c \geq \frac{1}{2\text{Re}(\lambda_{\min}(H_s))}$ and feedback gain matrix $K = B^T Q$. This completes the proof. □

B. Distributed Observer-based Output Feedback Control Protocol

Based on the measurement of the outputs of the neighbouring agents, the distributed observer-based output feedback protocol designed for the MAS (1) and (2) is as follows,

$$\begin{aligned} \dot{z}_i(t) &= Az_i(t) + Bu_i(t) - P(y_i(t) - Cz_i(t)) \\ u_i(t) &= -\overline{c}\overline{K}(\sum_{j\in\mathcal{N}_i} |a_{ij}|(z_i(t) - \textit{sign}(a_{ij})z_j(t)) \\ &\quad + \delta_i(z_i(t) - s_i z_0(t-\Delta))) \\ \dot{z}_0(t) &= Az_0(t) - P(y_0(t) - Cz_0(t)) \end{aligned} \tag{12}$$

where $z_i(t)$ and $z_0(t)$ are the state-estimates of the i^{th} follower and the leader, respectively, and $P \in \mathbb{R}^{n\times m}$ is observer gain matrix, chosen such that $(A + PC)$ is a Hurwitz matrix. Such a gain matrix P exists because (A, C) is detectable. Moreover, $\overline{c} \in \mathbb{R}$ and $\overline{K} \in \mathbb{K}^{p\times n}$ are design gain parameters in the form of scalar coupling and feedback matrix, respectively. A Theorem is presented below to realize the lag-bipartite consensus with the distributed output feedback control protocol.

Theorem 2. *MAS* (1) *and* (2) *achieve lag-bipartite consensus with distributed observer-based output feedback control law* (12) *over a structurally balanced signed digraph if the* *designed parameters are such that* $\overline{c} \geq \frac{1}{2Re(\lambda_{min}(H_s))}$ *and* $\overline{K} = B^T Q_2$ *where* $W = Q_2^{-1} \in \mathbb{R}^{n\times n}$ *is the positive definite solution to the following algebraic Riccati inequality (ARI):*

$$WA^T + AW - BB^T < 0 \tag{13}$$

Proof. Let, $e_i(t) = \begin{bmatrix} e_{x_i}(t) \\ e_{z_i}(t) \end{bmatrix} = \begin{bmatrix} x_i(t) - s_i x_0(t-\Delta) \\ z_i(t) - s_i z_0(t-\Delta) \end{bmatrix} \in \mathbb{R}^{2n}$, $(\forall i \in \Im_N)$ be the extended error vector, where $e_{x_i}(t) \in \mathbb{R}^n$ is the lag-bipartite consensus error and $e_{z_i}(t) \in \mathbb{R}^n$ is the observer estimation error for i^{th} agent. Then, using (1), (2), and (12), extended error state for i^{th} agent can be given as,

$$\dot{e}_i(t) = \overline{A}e_i(t) - \overline{c}\Gamma\Big(\sum_{j\in\mathcal{N}_i} |a_{ij}|(e_i(t) - sign(a_{ij})e_j(t)) + \delta_i e_i(t)\Big) \tag{14}$$

where $\overline{A} = \begin{bmatrix} A & \mathbf{0} \\ -PC & A+PC \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \mathbf{0} & B\overline{K} \\ \mathbf{0} & B\overline{K} \end{bmatrix}$. Then, in a more compact form, (14) can be written as follows,

$$\dot{e}(t) = (I_N \otimes \overline{A} - \overline{c}H_s \otimes \Gamma)e(t) \tag{15}$$

where $e(t) = [e_1^T(t)\ e_2^T(t) \dots e_N^T(t)]^T \in \mathbb{R}^{2Nn}$ is the stacked extended error vector. Obviously, (15) is a stability problem.

Let, $\zeta(t) = (I_N \otimes \Omega)e(t)$ where $\Omega = \begin{bmatrix} I_n & -I_n \\ \mathbf{0} & I_n \end{bmatrix}$. It can be easily obtained that $\Omega^{-1} = \begin{bmatrix} I_n & I_n \\ \mathbf{0} & I_n \end{bmatrix}$. Therefore, the following can be obtained,

$$\dot{\zeta}(t) = (I_N \otimes \widehat{A} - \overline{c}H_s \otimes \widehat{\Gamma})\zeta(t) \tag{16}$$

where $\widehat{A} = \Omega\overline{A}\Omega^{-1} = \begin{bmatrix} A+PC & \mathbf{0} \\ -PC & A \end{bmatrix}$, and $\widehat{\Gamma} = \Omega\Gamma\Omega^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B\overline{K} \end{bmatrix}$. Clearly, (16) is also a stability problem. Therefore, following the same methods as in Theorem 1, a new extended error dynamics can be obtained as,

$$\dot{\omega}(t) = (I_N \otimes \widehat{A} - \overline{c}J \otimes \widehat{\Gamma})\omega(t) \tag{17}$$

where $\omega(t) = (\Sigma^{-1} \otimes I_{2n})\zeta(t) \in \mathbb{R}^{2Nn}$ is the new extended error state, and $J = \Sigma^{-1}H_s\Sigma$ is the Jordan form of H_s. Clearly, lag-bipartite consensus is reached with asymptotic stabilization of the observer estimation error if the matrix $(I_N \otimes \widehat{A} - \overline{c}J \otimes \widehat{\Gamma})$ is Hurwitz. Moreover, $(I_N \otimes \widehat{A} - \overline{c}J \otimes \widehat{\Gamma})$ is an upper triangular block matrix whose block diagonal elements are $(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma})$, and therefore, the eigenvalues of $(I_N \otimes \widehat{A} - \overline{c}J \otimes \widehat{\Gamma})$ are given by the eigenvalues of $(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma})$. Consequently, if $(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma})$ $(i \in \Im_N)$ is Hurwitz, then lag-bipartite consensus is reached with asymptotic stabilization of observer estimation error. Let, $\overline{Q} = \text{diag}(Q_1, Q_2)$ where $Q_1 = Q_1^T \in \mathbb{R}^{n\times n}$ satisfies $(A+PC)^T Q_1 + Q_1(A+PC) < 0$ for the Hurwitz matrix $(A+PC)$, and Q_2 satisfies (13). Then, from Lyapunov stability theorem, for Hurwitz matrix $(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma})$, one obtains the following,

$$(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma})^T\overline{Q} + \overline{Q}(\widehat{A} - \overline{c}\lambda_i(H_s)\widehat{\Gamma}) < 0 \tag{18}$$

Moreover, (18) can be equivalently written as,

$$\begin{bmatrix} \Upsilon & -\chi \\ -\chi^T & A^TQ_2 + Q_2A - 2\overline{c}\lambda_i(H_s)Q_2B\overline{K} \end{bmatrix} < 0 \tag{19}$$

where $\chi = C^TP^TQ_2 \in \mathbb{R}^{n\times n}$, $\Upsilon \in \mathbb{R}^{n\times n}$ and $\Upsilon = (A+PC)^TQ_1 + Q_1(A+PC) < 0$. On pre- and post-multiplication of (19) by $\text{diag}(I, Q_2^{-1})$ is yielded,

$$\begin{bmatrix} \Upsilon & -C^TP^T \\ -PC & WA^T + AW - 2\overline{c}\lambda_i(H_s)BB^T \end{bmatrix} < 0 \tag{20}$$

where $W = Q_2^{-1}$ and $\overline{K} = B^TQ_2$. Since, $\Upsilon < 0$, therefore, from Schur complement lemma [21], it can be easily obtained that $WA^T + AW - 2\overline{c}\lambda_i(H_s)BB^T < 0$. Then, considering $\overline{c} \geq \frac{1}{2\text{Re}(\lambda_{\min}(H_s))}$, and $\overline{K} = B^TQ_2$, ARI (13) can be obtained. This completes the proof. □

V. Example and Simulations

Fig. 1 shows the signed digraph of a group of seven followers and a leader. Clearly, the leader, indicated by the 0^{th} node, is the root of the rooted directed spanning tree of the graph topology. The cooperative and the competitive interactions between the agents are represented by the solid and the dotted lines, respectively. The structural balance property of the underlying communication topology is evident from Fig. 1 with $\mathcal{V}_1 = \{1,2,3,4\}$ and $\mathcal{V}_2 = \{5,6,7\}$. In addition, agent 1 and agent 5 are cooperative and competitive with the leader, respectively. Various matrices for the MAS are: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Obviously, (A, B) is stabilizable and (A, C) is detectable. Moreover, observer gain matrix $P = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ is chosen such that $(A+PC)$ is Hurwitz. Moreover, scalar coupling gains $c = 3.0274$ and $\overline{c} = 16.0274$ are chosen from Theorem 1 and Theorem 2, respectively. Using MATLAB [22], for Theorem 1, ARE (6) solution $Q = \begin{bmatrix} 1.9123 & 0.4142 \\ 0.4142 & 1.3522 \end{bmatrix}$ and feedback gain matrix $K = \begin{bmatrix} 0.4142 & 1.3522 \end{bmatrix}$ are obtained, while for Theorem 2, ARI (13) solution $W = \begin{bmatrix} 10.3300 & -0.5387 \\ -0.5387 & 10.3300 \end{bmatrix}$, and feedback gain matrix $\overline{K} = -B^TQ_2 = -B^TW^{-1} = \begin{bmatrix} -0.0811 & -1.5558 \end{bmatrix}$ are obtained. The initial state values $x_i(0)$ $(i = 0, 1, \dots, 7)$ of i^{th} agent are: $x_0(0) = [0.34\ \ 0.64]^T$, $x_1(0) = [-0.10\ \ 0.14]^T$, $x_2(0) = [0.45\ \ -0.14]^T$, $x_3(0) = [-0.18\ \ -0.64]^T$, $x_4(0) = [0.50\ \ 0.60]^T$, $x_5(0) = [-0.15\ \ -0.18]^T$, $x_6(0) = [-0.56\ \ 0.78]^T$, and $x_7(0) = [0.60\ \ -0.14]^T$. The initial observer-state values

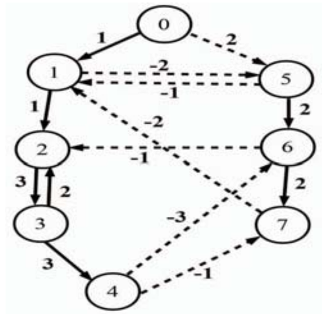

Fig. 1. Communication graph topology of the MAS

$z_i(0)$ $(i = 0, 1, \ldots, 7)$ for i^{th} agent's observer are: $z_0(0) = [0.24 \quad 0.74]^T$, $z_1(0) = [0.40 \quad -0.34]^T$, $z_2(0) = [-0.14 \quad -0.16]^T$, $z_3(0) = [-0.38 \quad 0.42]^T$, $z_4(0) = [0.30 \quad -0.56]^T$, $z_5(0) = [-0.25 \quad -0.28]^T$, $z_6(0) = [-0.60 \quad 0.10]^T$, and $z_7(0) = [0.50 \quad -0.42]^T$. The lag factor is taken as $\Delta = 2$ sec.

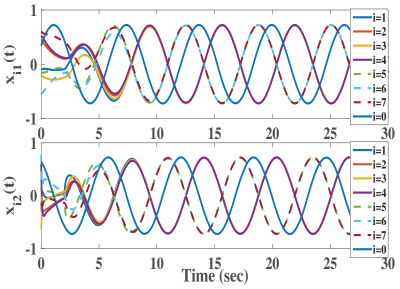

Fig. 2. State evolution $x_{i1}(t)$ and x_{i2} (t) $(i = 0, 1, \ldots, 7)$ of MAS with state feedback law (5) and with Δ= 2 sec

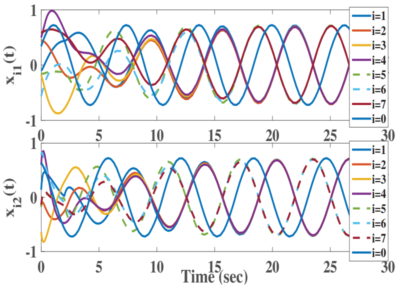

Fig. 3. State evolution $x_{i1}(t)$ and $x_{i2}(t)$ $(i = 0, 1, \ldots, 7)$ of MAS with output feedback law (12) and with Δ= 2 sec

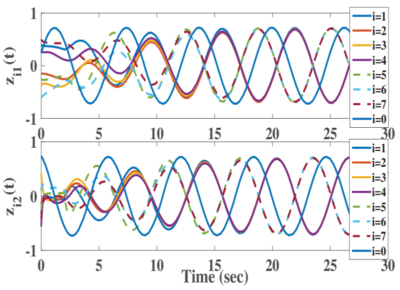

Fig. 4. Observer state evolution $z_{i1}(t)$ and $z_{i2}(t)$ $(i = 0, 1, \ldots, 7)$ of MAS with output feedback law (12) and with Δ= 2 sec

The state evolution of MAS (1) and (2) with state feedback control protocol (5) is shown Fig. 2. It can be easily seen that agents in $\mathcal{V}_1$ reach consensus with the leader state value in both sign and magnitude after a delay of $\Delta = 2$ sec while agents in $\mathcal{V}_2$ reach consensus in modulus with the leader state value, opposite in sign after the delay $\Delta = 2$ sec. Thus, bipartite consensus is achieved for the two groups of agents while with respect to the leader, they reach lag-bipartite consensus. On the other hand, Fig. 3 and 4 show the state and observer state evolutions with output feedback control protocol (12) where also it is observed that lag-bipartite consensus is achieved with a delay of $\Delta = 2$ sec that further demonstrates that both lag-bipartite consensus and estimation error are simultaneously stabilized with the distributed output feedback control law (12).

VI. Conclusions

Lag-bipartite consensus of linear MAS has been investigated over a structurally balanced signed digraph containing a rooted directed spanning tree having its root at the leader. Two distributed feedback controllers with suitably designed gain parameters achieve lag-bipartite consensus. The numerical simulations corroborate the theoretical results. However, the consensus control protocols of this work depend on the spectral property of the communication network. The future direction of the research will be to implement fully adaptive control laws which are independent of topology information.

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