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Synchronization of N-Coupled Hindmarsh-Rose Neuron Model with Time-Varying Delays

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***Abstract*—This paper dwells upon the synchronization phenomena in Hindmarsh-Rose (HR) neuron model of N number of neurons with time-varying communication delays between them. By considering a suitable Lyapunov-Krasovskii (L-K) functional, the synchronization criteria is obtained in terms of linear matrix inequality (LMI) for asymptotically stabilizing the error dynamics between the master-slave neuron configuration. The effectiveness of the proposed method is established through the numerical simulations of the neuron model.**

Keywords*—Synchronization, time-varying delay, Hindmarsh-Rose neuron model, Lyapunov-Krasovskii (LK) functional, linear matrix inequality (LMI).

I. Introduction

Synchronization is one of the most important and significant properties of complex dynamical networks [1], [2]. Rigorous analysis on synchronization is provided in [3]. The pioneering work [4] laid the foundation in this direction. Neural synchronization have been studied in biological systems also [5], [6]. Abnormal neural synchronization is associated with various neurological disorders such as schizophrenia [7], Parkinson's disease [8]. Synchronization or consensus is also studied in multi-agent systems and robotics [9], [10] and secure communication and cryptography [11].

Various behaviours of neurons are based on dynamical principles. A detailed and rigorous analysis is provided in [12]. Hodgkin and Huxley (HH) in their seminal work [13] demonstrated the underlying mechanism of membrane potential based on ionic conductance. FitzHugh and Nagumo [14] simplified the four-state HH model to a two-state model extracting its excitation dynamics. The shortcoming of FitzHugh-Nagumo (FHN) model is its inability to exhibit bursting behaviour of neurons. This problem was overcome by Hindmarsh-Rose (HR) [15] with their three-state model that can exhibit various neuro-computational features [16].

In [17], synchronization condition of two electrically coupled HR neurons under low-energy cost is studied. Adaptive feedback control schemes are proposed in [18] and [19] to achieve synchronization between two HR neurons with mismatched parameters. Multistate and multistage synchronization of HR neuron in presence of both chemical and electrical synapse are analyzed in [20]. The stability analysis of the synchronizing states in the identical and non-identical HR neuronal network is provided in [21]. Both nearest and global coupling through membrane potential and spiking variable are considered in this work.

It is certain that, like in every-other communication, the existence of time-delay is inevitable in neuronal communication as well. Synchronization of neural networks and complex dynamical systems with time-delays has been analyzed in details in [1], [2], [5], [22]. In [23], constant time-delay synchronization of two HR neurons is achieved using a linear adaptive feedback controller and a parameter update law. The effect of constant time-delay on bursting frequency and phase synchronization in coupled neurons is studied in [24]. Suitable controllers are designed to achieve synchronization of constant time-delayed two-coupled HR neuron system with stochastic noise in [25]. Synchronization criteria for two-coupled HR neurons under state-dependent constant delay is analyzed in [26].

In this work, synchronization is studied for a network of N number of coupled HR neurons with time-varying communication delays between them.

The organization of the paper is as follows. In Section II, the model description is laid out followed by the proposed synchronization criteria in Section III. Section IV presents the simulated results and establishes the accuracy and effectiveness of the methods while Section V provides the concluding remarks.

II. Preliminaries and Model Description

Notations: $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R} = (\infty,-\infty)$ denote the set of all positive real numbers, and the set of all real numbers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional Euclidean space, and the set of all $m \times n$ real matrices, respectively. The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $A^T \in \mathbb{R}^{n\times m}$. $A \in \mathbb{R}^{n\times n} > 0$ ($A \in \mathbb{R}^{n\times n} < 0$) means that the matrix A is symmetric, positive (negative) definite while $A \in \mathbb{R}^{n\times n} \geq 0$ ($A \in \mathbb{R}^{n\times n} \leq 0$) means the matrix is symmetric, positive (negative) semidefinite. $I_n \in \mathbb{R}^{n\times n}$ represents a real identity matrix of $n \times n$ dimension. $A \otimes B$ implies the Kronecker product of two matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ to yield a matrix $\Theta \in \mathbb{R}^{pr\times qs}$. $||.||$ represents the norm of a vector. If there exists an interaction between i-th node and j-th node in the neuron network, then it is represented by $i \leftrightarrow j$, and if there exists no interaction between these nodes, then denoted by $i \not\leftrightarrow j$, where $i,\ j = 1,2,\ldots,N$, and N is the number of neurons in the network. A diagonal matrix is represented by $diag(g_1, g_2, \ldots, g_n) \in \mathbb{R}^{n\times n}$ with g_i being diagonal elements $(i = 1,2,\ldots,n)$.

The dynamical representation of an uncoupled HR neuron is given as below,

$$\begin{aligned}\dot{x}(t) &= y(t) - bx^3(t) + ax^2(t) - z(t) + I_{ext} \\ \dot{y}(t) &= c - dx^2(t) - y(t) \\ \dot{z}(t) &= r(s(x(t) - \bar{x}) - z(t)) \end{aligned} \tag{1}$$

where states x, y, and z represent the membrane potential, spiking variable (fast current), and bursting variable (slow adaptation current), and a, b, c, d, r, and s are real constant parameters while $\bar{x}$ represents the resting potential of the neuron (1), and I_{ext} is the externally applied current to the neuron.

In this work, a coupled network of N number of HR neurons with time-varying communication delays has been considered. A network model of coupled HR neurons with time-varying communication delays is as follows,

$$\begin{aligned}\dot{x}_i(t) &= y_i(t) - bx_i^3(t) + ax_i^2(t) - z_i(t) + I_{ext} \\ &\quad + \bar{c}\sum_{\substack{j=1 \\ j\neq i}}^{N} g_{ij}(x_j(t-\tau(t)) - x_i(t-\tau(t))) \\ \dot{y}_i(t) &= c - dx_i^2(t) - y_i(t) \\ \dot{z}_i(t) &= r(s(x_i(t) - \bar{x}) - z_i(t)) \end{aligned} \tag{2}$$

where $i,\ j = 1, 2, \ldots, N$ with N being the number of neurons in the network, $\bar{c} \in \mathbb{R}$ represents the overall coupling strength in the network, $G = (g_{ij})_{N\times N}$ represents the topology of the network and is known as coupling matrix whose elements are defined as below,

$$g_{ij} = \begin{cases} g_{ji} \geq 0, & \text{if } i \leftrightarrow j,\ \forall\ i \neq j \\ g_{ji} = 0, & \text{if } i \not\leftrightarrow j,\ \forall\ i \neq j \\ g_{ii} = -\sum\limits_{\substack{j=1 \\ j\neq i}}^{N} g_{ij}, & \forall i = j \end{cases} \tag{3}$$

Remark: It is evident from (3) that $G \in \mathbb{R}^{N\times N}$ is a row (and column) sum zero matrix.

If all the elements of the coupling matrix G is 1, then the HR neurons in the network are said to have uniform coupling. A model of coupled HR neurons with uniform coupling takes the form as

$$\begin{aligned}\dot{x}_i(t) &= y_i(t) - bx_i^3(t) + ax_i^2(t) - z_i(t) + I_{ext} \\ &\quad + \bar{c}\sum_{\substack{j=1 \\ j\neq i}}^{N} (x_j(t-\tau(t)) - x_i(t-\tau(t))) \\ \dot{y}_i(t) &= c - dx_i^2(t) - y_i(t) \\ \dot{z}_i(t) &= r(s(x_i(t) - \bar{x}) - z_i(t)) \end{aligned} \tag{4}$$

The master neuron model is considered to be an isolated (uncoupled) node and is given as,

$$\begin{aligned}\dot{x}_m(t) &= y_m(t) - bx_m^3(t) + ax_m^2 - z_m(t) + I_{ext} \\ \dot{y}_m(t) &= c - dx_m^2(t) - y_m(t) \\ \dot{z}_m(t) &= r(s(x_m(t) - \bar{x}) - z_m(t)) \end{aligned} \tag{5}$$

The master neuron model has the reference trajectory for other neurons to achieve synchronization. In the next section, the analysis of synchronization of the neurons is provided in details.

III. Synchronization of Coupled HR Neurons with Time-varying Delay

In this section, synchronization criteria for N number of coupled HR neurons under time-varying communication delays is derived.

Definition: Synchronization between i-th neuron and a master neuron are said to be achieved if

$$\lim_{t\to\infty} \|X_i(t) - X_m(t)\| = 0,\ \forall i = 1, 2, \ldots, N$$

where $X_i = [x_i^T,\ y_i^T,\ z_i^T]^T$ and $X_m = [x_m^T,\ y_m^T,\ z_m^T]^T$ represent the state vector of a i-th neuron and master neuron, respectively.

Assumption: The time varying delay $\tau(t)$ is considered such that $\dot{\tau}(t) < \sigma$ where $\sigma > 0$.

The neuron model (2) augmented by the control laws w_{ix}, w_{iy}, and w_{iz} ($i = 1, 2, \ldots, N$) is considered the slave model and is given as below,

$$\begin{aligned}\dot{x}_{is}(t) &= y_{is}(t) - bx_{is}^3(t) + ax_{is}^2(t) - z_{is}(t) + I_{ext} \\ &\quad + \bar{c}\sum_{\substack{j=1 \\ j\neq i}}^{N} g_{ij}(x_{js}(t-\tau(t)) - x_{is}(t-\tau(t))) \\ &\quad + w_{ix}(t) \\ \dot{y}_{is}(t) &= c - dx_{is}^2(t) - y_{is}(t) + w_{iy}(t) \\ \dot{z}_{is}(t) &= r(s(x_{is}(t) - \bar{x}) - z_{is}(t)) + w_{iz}(t) \end{aligned} \tag{6}$$

where the control laws are given as,

$$\begin{aligned} w_{ix}(t) &= b(x_{is}^3(t) - x_m^3(t)) - a(x_{is}^2(t) - x_m^2(t)) \\ &\quad + k_1(x_{is}(t) - x_m(t)) \\ &\quad + k_2(x_{is}(t-\tau(t)) - x_m(t-\tau(t))) \\ w_{iy}(t) &= d(x_{is}^2(t) - x_m^2(t)) + k_3(y_{is}(t) - y_m(t)) \\ &\quad + k_4(y_{is}(t-\tau(t)) - y_m(t-\tau(t))) \\ w_{iz}(t) &= k_5(z_{is}(t) - z_m(t)) \\ &\quad + k_6(z_{is}(t-\tau(t)) - z_m(t-\tau(t))) \end{aligned} \tag{7}$$

with k_i $(i = 1, 2, \ldots, 6)$ being the controller gains in (7).

The errors between the corresponding states of the slave model (6) and master model (5) are defined as $e_{i1}(t) = x_{is}(t) - x_m(t)$, $e_{i2}(t) = y_{is}(t) - y_m(t)$, and $e_{i3}(t) = z_{is}(t) - z_m(t)$ ($i = 1, 2, \ldots, N$). Thus, the error dynamics for i-th neuron can be written as,

$$\begin{aligned}\dot{e}_{i1}(t) &= e_{i2}(t) - e_{i3}(t) + k_1e_{i1}(t) + k_2e_{i1}(t-\tau(t)) \\ &\quad + \bar{c}\sum_{j=1}^{N} g_{ij}e_{j1}(t-\tau(t)) \\ \dot{e}_{i2}(t) &= (k_3 - 1)e_{i2}(t) + k_4e_{i2}(t-\tau(t)) \\ \dot{e}_{i3}(t) &= rse_{i1}(t) + (k_5 - r)e_{i3}(t) + k_6e_{i3}(t-\tau(t)) \end{aligned} \tag{8}$$

If the error vector and the delayed error vector for the i-th neuron are defined as $e_i(t) = [e_{i1}^T(t),\ e_{i2}^T(t),\ e_{i3}^T(t)]^T$, and $e_i(t-\tau(t)) = [e_{i1}^T(t-\tau(t)),\ e_{i2}^T(t-\tau(t)),\ e_{i3}^T(t-\tau(t))]^T$, respectively, then (8) can be written in a compact form as,

$$\dot{e}_i(t) = (\Sigma + \Delta_1)e_i(t) + \Delta_2e_i(t-\tau(t)) + \bar{c}\sum_{j=1}^{N} g_{ij}Ae_j(t-\tau(t)) \tag{9}$$

where $\Sigma = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ rs & 0 & -r \end{bmatrix}$, $\Delta_1 = \text{diag}[k_1, k_3, k_5]$, $\Delta_2 = \text{diag}[k_2, k_4, k_6]$, and $A = diag(1, 0, 0)$.

The error dynamics of the network of N number of HR neurons is derived as,

$$\dot{e}(t) = [I_N \otimes (\Sigma + \Delta_1)]e(t) + [(I_N \otimes \Delta_2) + \bar{c}(G \otimes A)] \times e(t - \tau(t)) \quad (10)$$

where the error vector of the system is given by,

$$e(t) = [e_1^T(t),\ e_2^T(t), \dots,\ e_N^T(t)]^T.$$

The objective is to obtain the synchronization criteria for the coupled HR neurons with time-varying communication delays between them. This is achieved when $\lim_{t\to\infty} \|e(t)\| = 0$.

The following theorem is proposed to achieve the synchronization of the HR neurons.

Theorem 1: Given scalers $\bar{c} < 0$ and $\sigma > 0$, a network of HR neurons with time-varying delay given by (6) synchronizes if there exist a diagonal matrix $\bar{R} \in \mathbb{R}^{3N\times 3N} > 0$, $H \in \mathbb{R}^{3N\times 3N}$ where $H = diag(H_1, H_2, \dots, H_N)$ and $H_i = diag(h_{i1}, h_{i2}, h_{i3})$ such that $h_{il} > 0$ (i=1,2,…,N and l=1,2,3) and a diagonal matrix $M \in \mathbb{R}^{3N\times 3N} > 0$ such that the following LMI holds

$$\begin{bmatrix} \Phi_1 & \Phi_2 \\ \star & (\sigma - 1)MH \end{bmatrix} \leq 0 \quad (11)$$

where $\Phi_1 = M\bar{R}[I_N \otimes (\Sigma + \Delta_1)] + [I_N \otimes (\Sigma^T + \Delta_1^T)]\bar{R}M + MH$, and $\Phi_2 = M\bar{R}(I_N \otimes \Delta_2) + \bar{c}M\bar{R}(G \otimes A)$

Proof: A suitable L-K functional is chosen as below,

$$V(e(t)) = e^T\bar{R}e(t) + \sum_{i=1}^{N}\sum_{l=1}^{3}\int_{t-\tau(t)}^{t} h_{il}e_{il}^2(s)\,\mathrm{d}s \quad (12)$$

Time-derivative of (12) along the trajectories of (10) yields,

$$\begin{aligned}\dot{V}(e(t)) &= 2e^T(t)\bar{R}\dot{e}(t) + \sum_{i=1}^{N}\sum_{l=1}^{3}[h_{il}e_{il}^2(t) - h_{il}e_{il}^2(t-\tau(t)) \\ &\quad \times (1 - \dot{\tau}(t))] \\ &\leq 2e^T(t)\bar{R}[I_N \otimes (\Sigma + \Delta_1)]e(t) + 2e^T(t)\bar{R}[(I_N \otimes \Delta_2) \\ &\quad + \bar{c}(G \otimes A)]e(t-\tau(t)) + \sum_{i=1}^{N}\sum_{l=1}^{3}[h_{il}e_{il}^2(t) \\ &\quad - h_{il}e_{il}^2(t-\tau(t))(1-\sigma)] \\ &= \psi^T \begin{bmatrix} \bar{\Phi}_1 & \bar{\Phi}_2 \\ \star & (\sigma-1)H \end{bmatrix} \psi \end{aligned} \quad (13)$$

where $\psi = \begin{bmatrix} e(t) \\ e(t-\tau(t)) \end{bmatrix}$, $\bar{\Phi}_1 = 2\bar{R}[I_N \otimes (\Sigma + \Delta_1)] + H$, and $\bar{\Phi}_2 = \bar{R}[(I_N \otimes \Delta_2) + \bar{c}(G \otimes A)]$.

To achieve synchronization among the neuron states i.e., stabilizing the error dynamics, $\dot{V}(e(t)) \leq 0$ which implies

$$\begin{bmatrix} \bar{\Phi}_1 & \bar{\Phi}_2 \\ \star & (\sigma - 1)H \end{bmatrix} \leq 0 \quad (14)$$

It can be easily observed that $\bar{\Phi}_1$ in (14) is not a symmetric matrix. Pre-multiplying (14) by $\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ where $M \in \mathbb{R}^{3N\times 3N} > 0$ for symmetrizing yields

$$\begin{bmatrix} \Phi_1 & \Phi_2 \\ \star & (\sigma - 1)MH \end{bmatrix} \leq 0 \quad (15)$$

which completes the proof. ■

IV. Simulation and Results

In this work, $N = 4$ number of coupled HR neurons, and an isolated master neuron is considered. All the simulations have been performed in MATLAB [27]. By choosing the following set of parameters for an isolated master neuron, the simulation of the trajectory of the uncoupled neuron model (1) is shown in Fig. 1.

$a = 3$, $b = 1$, $c = 1$, $d = 5$, $\bar{x} = 1.6$, $s = 4$, $I_{ext} = 3.1$, $r = 0.005$, $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = 0.3$ where $x(0)$, $y(0)$, and $z(0)$ are the initial state values of x, y and z in (1).

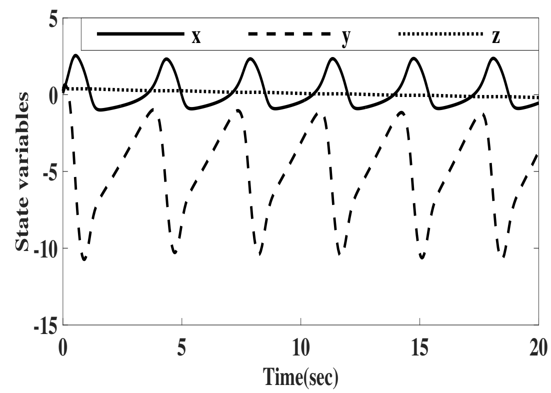

Fig. 1: Time evolution of uncoupled HR neuron

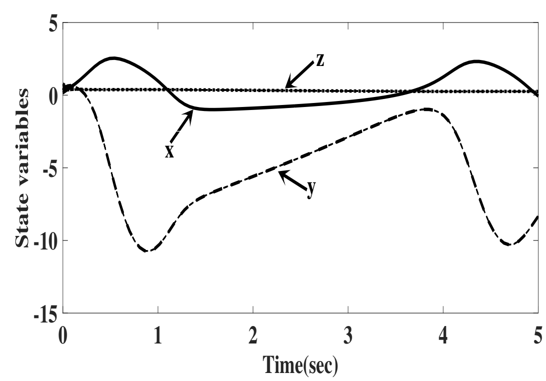

Fig. 2: Synchronization of four coupled HR neurons and master neuron with the proposed control law

In addition to the above parameters, to achieve synchronization with the isolated master neuron, and hence synchronization among the $N = 4$ neurons (2), the following parameters are also considered: $\tau(t) = 1 - 0.7e^{-t}$, $\sigma = 2$, $\bar{c} = -0.2$,

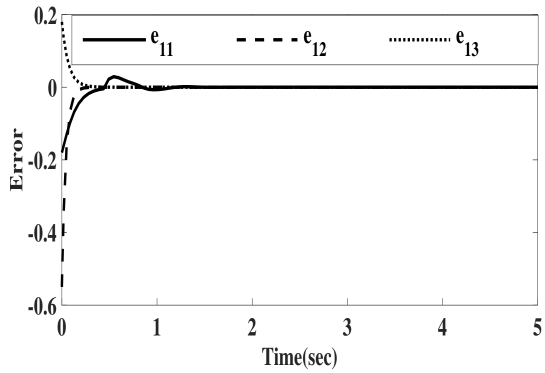

Fig. 3: Error trajectories of neuron 1 and master neuron

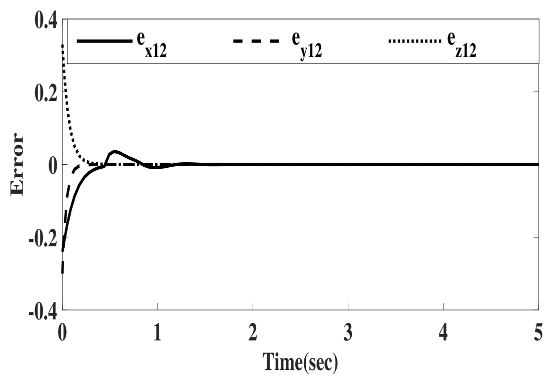

Fig. 4: Error trajectories of neuron 1 and neuron 2

$X_{10} = [0.1 \quad 0.24 \quad 0.56]^T$, $X_{20} = [0.34 \quad 0.54 \quad 0.23]^T$, $X_{30} = [0.13 \quad 0.22 \quad 0.64]^T$, $X_{40} = [0.18 \quad 0.34 \quad 0.78]^T$, and $X_{m0} = [0.28 \quad 0.79 \quad 0.38]^T$ where $X_{i'0}$ $(i' = 1, 2, 3, 4)$ are the initial states of i'-th neuron, and X_{m0} is the initial state of the master neuron. The coupling matrix of the network (2) is considered as,

$$G = \begin{bmatrix} -1.5 & 0.6 & 0.5 & 0.4 \\ 0.6 & -0.8 & 0.1 & 0.1 \\ 0.5 & 0.1 & -3.1 & 2.5 \\ 0.4 & 0.1 & 2.5 & -3.0 \end{bmatrix} \tag{16}$$

The following gain matrices are obtained by solving LMI in (11) using MATLAB YALMIP toolbox [28] and coupling matrix given in (16):

$$\Delta_1 = \begin{bmatrix} -9.3513 & 0 & 0 \\ 0 & -23.6313 & 0 \\ 0 & 0 & -14.7192 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} -4.5523 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.0002 \end{bmatrix}$$

Fig. 2 shows the evolution trajectories of the four HR neurons, and the isolated master neuron with the proposed control laws (7). It is observed that they are synchronized under the control laws (7). Fig. 3 shows the error dynamics between neuron 1 and the master neuron where e_{11}, e_{12}, and e_{13} represent the errors between the corresponding states of neuron 1 and master neuron, respectively. It is observed that the errors between the corresponding states are asymptotically stable, reaching a zero steady-state value. Fig. 4 shows the error dynamics between neuron 1 and neuron 2 where e_{x12}, e_{y12}, and e_{z12} represent the errors between corresponding states of neuron 1 and neuron 2, respectively. It can be observed that the respective errors between the corresponding states of neuron 1 and neuron 2 are asymptotically stable, reaching a zero steady-state value. These results show that with the proposed control laws, the synchonization of four coupled HR neurons has been achieved under time-varying delays.

V. Conclusion

This work investigates synchronization criteria for a network of N number of HR neuron model. The $N = 4$ coupled network achieves synchronization under time-varying delay with different initial conditions. The synchronization criteria is given in terms of LMI. As a future scope, the present design framework can be used for other neuron models with the effect of stochastic noise.

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